

Stabilization of Kelvin-Voigt viscoelastic fluid flow model

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Abstract

In this article, stabilization result for the viscoelastic fluid flow problem governed by Kelvin-Voigt model, that is, convergence of the unsteady solution to its steady state solution is proved under the assumption that linearized self-adjoint steady state eigenvalue problem has a minimal positive eigenvalue. Both the power and exponential convergence of the unsteady solution to the steady state solution is proved under various prescribed conditions on the forcing function. It is shown that results are valid uniformly in κ as $\kappa \rightarrow 0$.

1 Introduction

Consider the following equation arising in Kelvin-Voigt model of viscoelastic fluid flow problem: Find (\mathbf{u}, p) such that

$$(1.1) \quad \mathbf{u}_t - \kappa \Delta \mathbf{u}_t - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}(x, t) \quad x \in \Omega, \quad t > 0,$$

$$(1.2) \quad \nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \quad t > 0,$$

$$(1.3) \quad \mathbf{u}(x, 0) = u_0 \quad x \in \Omega, \quad \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega, \quad t \geq 0,$$

where, $\nu > 0$ is the coefficient of kinematic viscosity, $\kappa > 0$ is the retardation time or the time of relaxation of deformations and Ω is a bounded convex polygonal or polyhedral domain in \mathbb{R}^d , $d = 2, 3$ with boundary $\partial\Omega$. Since the system differs from the system of Navier-Stokes equations by $-\kappa \Delta \mathbf{u}_t$, then one is curious to explore how far results on stabilization carry over to the Kelvin-Voigt model (1.1)-(1.3). Therefore, in this article, both power and exponential convergence of the unsteady solution to the steady state solution is proved under various assumptions on the forcing function $\mathbf{f}(x, t)$.

Regarding viscoelastic fluid flow problem, Pavlovskii [13] first introduced this model as a model of weakly concentrated water-polymer mixture. Then, Oskolkov [8] and his collaborators called it as Kelvin-Voigt model. For applications of such models, see [1], [2] and [3] and reference, therein. For local and global solvability of the problem (1.1)-(1.3), we refer to [9], [10], [11], [12].

On stabilizability, Sobolevskii [14] has shown exponential convergence for the Oldroyd's model under the assumption that forcing function is Hölder continuous and exponentially decaying. Further, He *et al.* [6] has shown both exponential and power convergence for the solution by relaxing the Hölder continuity of the forcing function and assuming that forcing function \mathbf{f} has exponential or power decay property. For linearized viscoelastic flow problem asymptotic behavior is discussed in [5].

The main contribution of this article is on the convergence of the unsteady solution to its steady state solution. Further, both exponential and power convergence results are shown for the velocity and the pressure in various norms under a variety of assumptions on the forcing function. Moreover, it is proved that results are valid uniformly in κ as $\kappa \rightarrow 0$.

For the rest of this article, first we introduce \mathbb{R}^d , ($d = 2, 3$)- valued function denoted by bold face type letters as

$$\mathbf{H}_0^1 = (H_0^1(\Omega))^d, \quad \mathbf{L}^2 = (L^2(\Omega))^d \quad \text{and} \quad \mathbf{H}^m = (H^m(\Omega))^d,$$

where $L^2(\Omega)$ is the space of square integrable functions defined in Ω with inner product $(\phi, \psi) = \int_{\Omega} \phi(x)\psi(x) dx$ and norm $\|\phi\| = \left(\int_{\Omega} |\phi(x)|^2 dx \right)^{\frac{1}{2}}$. Further, $H^m(\Omega)$ denotes the standard Hilbert Sobolev space of order $m \in \mathbb{N}^+$ with norm $\|\phi\|_m = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha \phi|^2 dx \right)^{1/2}$. Note that \mathbf{H}_0^1 is equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^d (\partial_j v_i, \partial_j v_i) \right)^{1/2} = \left(\sum_{i=1}^d (\nabla v_i, \nabla v_i) \right)^{1/2}.$$

Let \mathbf{H}^{-1} be the dual space of \mathbf{H}_0^1 with norm $\|\cdot\|_{-1}$. For more details see [7].

The rest of the article is organized as follows. Section 2 focuses on the corresponding steady state problem with some properties. Section 3 is devoted to both exponential and power convergence result of unsteady solution to the corresponding steady state solution.

2 Steady state problem and its properties

In this section, first we introduce some spaces :

$$\mathbf{J}_1 = \{\phi \in \mathbf{H}_0^1 : \nabla \cdot \phi = 0\},$$

$$\mathbf{J} = \{\phi \in \mathbf{L}^2 : \nabla \cdot \phi = 0 \in \Omega, \quad \phi \cdot n = 0 \quad \text{on} \quad \partial\Omega\}$$

and H^m/\mathbb{R} be the quotient space with norm $\|p\|_{H^m/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|p + c\|_m$.

Setting $-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \rightarrow \mathbf{J}$ as the stokes operator. Note that the following Poincarè inequality holds true:

$$(2.1) \quad \|\mathbf{v}\|^2 \leq \frac{1}{\lambda_1} \|\nabla \mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{H}_0^1,$$

where λ_1^{-1} is the best possible constant depending on the domain Ω . Moreover, the following holds:

$$(2.2) \quad \|\nabla v\|^2 \leq \frac{1}{\lambda_1} \|\tilde{\Delta} \mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2.$$

The continuous bilinear forms $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$ on $\mathbf{H}_0^1 \times \mathbf{H}_0^1$ and $\mathbf{H}_0^1 \times L^2/\mathbb{R}$ are

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1,$$

$$d(\mathbf{v}, q) = -(\mathbf{v}, \nabla q) = (q, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1, q \in L^2/\mathbb{R},$$

and define the trilinear form $b(\cdot, \cdot, \cdot)$ as

$$b(\mathbf{v}, \mathbf{w}, \phi) := \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \phi) - \frac{1}{2}(\mathbf{v} \cdot \nabla \phi, \mathbf{w}), \quad \mathbf{v}, \mathbf{w}, \phi \in \mathbf{H}_0^1.$$

Note that for $\mathbf{v} \in \mathbf{J}_1$, $\mathbf{w}, \phi \in \mathbf{H}_0^1$; $b(\mathbf{v}, \mathbf{w}, \phi) = (\mathbf{v} \cdot \nabla \mathbf{w}, \phi)$.

The continuous bilinear form $d(\mathbf{v}, q)$ satisfy the property:

$$(2.3) \quad c\|q\| \leq \sup_{\mathbf{v} \in \mathbf{H}_0^1} \frac{d(\mathbf{v}, q)}{\|\nabla \mathbf{v}\|}, \quad \forall q \in L^2/\mathbb{R}.$$

The trilinear form satisfy the following properties:

$$(2.4) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1,$$

$$(2.5) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq N \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\| \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1,$$

$$(2.6) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq N \|\mathbf{u}\|^{1/2} \|\nabla \mathbf{u}\|^{1/2} \|\nabla \mathbf{v}\|^{1/2} \|\Delta \mathbf{v}\|^{1/2} \|\mathbf{w}\| \quad \forall \mathbf{u} \in \mathbf{H}_0^1, \mathbf{v} \in \mathbf{H}^2 \cap \mathbf{H}_0^1, \mathbf{w} \in \mathbf{L}^2;$$

$$(2.7) \quad b(\mathbf{v}, \mathbf{w}, \mathbf{w}) = 0 \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{J}_1.$$

For more details, we refer to [4] and [15].

An equilibrium (steady state) solution $(\mathbf{u}^\infty, p^\infty)$ to the continuous problem (1.1)-(1.3) satisfies

$$(2.8) \quad -\nu \Delta \mathbf{u}^\infty + \mathbf{u}^\infty \cdot \nabla \mathbf{u}^\infty + \nabla p^\infty = \mathbf{f}^\infty \quad \text{in } \Omega$$

$$(2.9) \quad \nabla \cdot \mathbf{u}^\infty = 0 \quad \text{in } \Omega$$

$$(2.10) \quad \mathbf{u}^\infty = 0 \quad \text{on } \partial\Omega,$$

where $\mathbf{f}^\infty = \lim_{t \rightarrow \infty} \mathbf{f}(t, x)$. In its weak form, the steady state solution satisfies $(\mathbf{u}^\infty, p^\infty) \in \mathbf{H}_0^1 \times L^2/\mathbb{R}$

$$(2.11) \quad \nu(\nabla \mathbf{u}^\infty, \nabla \phi) + (\mathbf{u}^\infty \cdot \nabla \mathbf{u}^\infty, \phi) + (\nabla p^\infty, \phi) = (\mathbf{f}^\infty, \phi) \quad \forall \phi \in \mathbf{H}_0^1,$$

$$(2.12) \quad (\nabla \cdot \mathbf{u}^\infty, \chi) = 0 \quad \forall \chi \in \mathbf{L}^2.$$

Equivalently, seek $\mathbf{u}^\infty \in \mathbf{J}_1$ such that

$$(2.13) \quad \nu(\nabla \mathbf{u}^\infty, \nabla \phi) + (\mathbf{u}^\infty \cdot \nabla \mathbf{u}^\infty, \phi) = (\mathbf{f}^\infty, \phi) \quad \forall \phi \in \mathbf{J}_1.$$

Now, we make the following assumption:

(A1) The eigenvalue problem

$$(2.14) \quad \begin{aligned} -\nu \Delta \bar{\mathbf{z}} + \frac{1}{2} [\nabla \mathbf{u}^\infty + (\nabla \mathbf{u}^\infty)^*] \bar{\mathbf{z}} + \bar{\nabla} q &= \lambda \bar{\mathbf{z}} \\ \nabla \cdot \bar{\mathbf{z}} &= 0 \\ \bar{\mathbf{z}} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has a minimum eigenvalue $\lambda_0 > 0$. This is a sufficient condition for a unique solvability of the problem (2.8)-(2.10).

Multiply (2.14) with $\bar{\mathbf{z}} \in \mathbf{H}^2 \cap \mathbf{H}_0^1$ with $\nabla \cdot \bar{\mathbf{z}} = 0$ to obtain

$$(2.15) \quad \nu \|\nabla \bar{\mathbf{z}}\|^2 + (\bar{\mathbf{z}} \cdot \nabla \mathbf{u}^\infty, \bar{\mathbf{z}}) = \lambda \|\bar{\mathbf{z}}\|^2.$$

That is

$$\nu \|\nabla \bar{\mathbf{z}}\|^2 - (\bar{\mathbf{z}} \cdot \nabla \bar{\mathbf{z}}, \mathbf{u}^\infty) = \lambda \|\bar{\mathbf{z}}\|^2.$$

So it follows that from (2.15) and (2.4) the following inequality holds

$$(2.16) \quad \nu a(\bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{z}}, \bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{u}}^\infty, \bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{z}}, \bar{\mathbf{u}}^\infty, \bar{\mathbf{z}}) = \nu a(\bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{z}}, \bar{\mathbf{u}}^\infty, \bar{\mathbf{z}}) = \lambda \|\bar{\mathbf{z}}\|^2 \geq \lambda_0 \|\bar{\mathbf{z}}\|^2.$$

Also it can be proved that

$$(2.17) \quad \nu a(\bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{z}}, \bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{u}}^\infty, \bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{z}}, \bar{\mathbf{u}}^\infty, \bar{\mathbf{z}}) = \nu a(\bar{\mathbf{z}}, \bar{\mathbf{z}}) + b(\bar{\mathbf{z}}, \bar{\mathbf{u}}^\infty, \bar{\mathbf{z}}) \geq \gamma_1 \|\nabla \bar{\mathbf{z}}\|^2$$

holds for some constant $\gamma_1 > 0$. For a proof, see, [6] and [14]. Before proceeding further, we first discuss on some *a priori* bounds of the steady state solution, which are needed in the sequel.

Remark 2.1. (i) Choose $\phi = \mathbf{u}^\infty$ in (2.13) to obtain

$$\nu \|\nabla \mathbf{u}^\infty\|^2 = (\mathbf{f}^\infty, \mathbf{u}^\infty) \leq \|\mathbf{f}^\infty\|_{-1} \|\nabla \mathbf{u}^\infty\|.$$

Therefore, $\|\nabla \mathbf{u}^\infty\|$ is bounded by

$$(2.18) \quad \|\nabla \mathbf{u}^\infty\| \leq \frac{1}{\nu} \|\mathbf{f}^\infty\|_{-1}.$$

(ii) From Poincarè inequality, it follows that

$$(2.19) \quad \|\mathbf{u}^\infty\| \leq \frac{1}{\sqrt{\lambda_1}} \|\nabla \mathbf{u}^\infty\| \leq \frac{1}{\nu \sqrt{\lambda_1}} \|\mathbf{f}^\infty\|_{-1}.$$

Therefore, $\|\mathbf{u}^\infty\|$ is also bounded.

(ii) Choose $\phi = -\tilde{\Delta} \mathbf{u}^\infty$ in (2.13) to arrive at

$$\begin{aligned} \nu \|\tilde{\Delta} \mathbf{u}^\infty\|^2 &= (\mathbf{f}^\infty, -\tilde{\Delta} \mathbf{u}^\infty) + (\mathbf{u}^\infty \cdot \nabla \mathbf{u}^\infty, \tilde{\Delta} \mathbf{u}^\infty) \\ &\leq \frac{\nu}{2} \|\tilde{\Delta} \mathbf{u}^\infty\|^2 + C(\nu) \|\mathbf{f}^\infty\|^2 + C(\nu) \|\mathbf{u}^\infty\|^2 \|\nabla \mathbf{u}^\infty\|^4, \end{aligned}$$

and

$$\nu \|\tilde{\Delta} \mathbf{u}^\infty\|^2 \leq C(\nu) (\|\mathbf{f}^\infty\|^2 + \|\mathbf{u}^\infty\|^2 \|\nabla \mathbf{u}^\infty\|^4).$$

Therefore, $\|\tilde{\Delta} \mathbf{u}^\infty\|$ is bounded.

(iv) From interpolation inequality in 2- dimension

$$(2.20) \quad \|\mathbf{u}^\infty\|_{L^\infty} \leq C \|\mathbf{u}^\infty\|^{1/2} \|\tilde{\Delta} \mathbf{u}^\infty\|^{1/2},$$

and therefore, $\|\mathbf{u}^\infty\|_{L^\infty}$ is also bounded.

(v) From Gagliardo-Nirenberg inequality in 2- dimension

$$\|\mathbf{u}^\infty\|_{L^4} \leq C \|\mathbf{u}^\infty\|^{1/2} \|\nabla \mathbf{u}^\infty\|^{1/2},$$

and therefore, $\|\mathbf{u}^\infty\|_{L^4}$ is bounded and similarly, $\|\nabla \mathbf{u}^\infty\|_{L^4}$ is also bounded.

3 Stabilization result

This section focuses on *a priori* bounds for the problem (3.4), which are valid uniformly in time using both power and exponential weight functions in time. It is, further, shown both the exponential and power convergence of $(\mathbf{u}(t), p(t))$ to $(\mathbf{u}^\infty, p^\infty)$, which is valid uniformly in κ as $\kappa \rightarrow 0$.

Let $\mathbf{z} = \mathbf{u} - \mathbf{u}^\infty$, $q = p - p^\infty$. Then, (\mathbf{z}, q) satisfy

$$(3.1) \quad \mathbf{z}_t - \kappa \Delta \mathbf{z}_t - \nu \Delta \mathbf{z} + \mathbf{z} \cdot \nabla \mathbf{z} + \mathbf{u}^\infty \cdot \nabla \mathbf{z} + \mathbf{z} \cdot \nabla \mathbf{u}^\infty + \nabla q = \mathbf{F},$$

$$(3.2) \quad \nabla \cdot \mathbf{z} = 0,$$

$$(3.3) \quad \mathbf{z}(x, 0) = \mathbf{u}_0 - \mathbf{u}^\infty = \mathbf{z}_0 \text{ (say)} \quad \text{and} \quad \mathbf{z} = 0 \quad \text{on} \quad \partial\Omega.$$

Now the weak formulation of (3.1)-(3.3) is to seek a pair of function $(\mathbf{z}(t), q(t)) \in \mathbf{H}_0^1 \times L^2/\mathbb{R}$ with $\mathbf{z}(0) = \mathbf{z}_0$ such that $\forall t > 0$

$$(3.4) \quad \begin{aligned} (\mathbf{z}_t, \phi) + \kappa a(\mathbf{z}_t, \phi) + \nu a(\mathbf{z}, \phi) + b(\mathbf{z}, \mathbf{z}, \phi) + b(\mathbf{u}^\infty, \mathbf{z}, \phi) + b(\mathbf{z}, \mathbf{u}^\infty, \phi) + (\nabla q, \phi) &= (\mathbf{F}, \phi) \quad \forall \phi \in \mathbf{H}_0^1, \\ (\nabla \cdot \mathbf{z}, \chi) &= 0 \quad \forall \chi \in L^2. \end{aligned}$$

Equivalently, find $\mathbf{z}(t) \in \mathbf{J}_1$ with $\mathbf{z}(0) = \mathbf{z}_0$ such that for $t > 0$

$$(3.5) \quad \begin{aligned} &(\mathbf{z}_t, \phi) + \kappa a(\mathbf{z}_t, \phi) + \nu a(\mathbf{z}, \phi) + b(\mathbf{z}, \mathbf{z}, \phi) + b(\mathbf{u}^\infty, \mathbf{z}, \phi) \\ &+ b(\mathbf{z}, \mathbf{u}^\infty, \phi) = (\mathbf{F}, \phi) \quad \forall \phi \in \mathbf{J}_1. \end{aligned}$$

Throughout this paper we always assume that

$$(3.6) \quad 0 < \alpha < \frac{\lambda_1}{4(1 + \lambda_1 \kappa)} \min \left\{ \nu, \gamma_1 \right\}, \quad \delta_0 > 0 \quad \text{and} \quad \alpha_1 = \alpha - \delta_0 > 0, \quad \beta = 2\delta.$$

$$\tau(t) = \max\{\bar{t}, t\}, \text{ where } \bar{t} = \frac{4\delta(1 + \kappa\lambda_1)}{\lambda_1} \max \left\{ \frac{1}{\nu}, \frac{1}{\gamma_1} \right\} \text{ if } \delta > 0 \quad \text{and} \quad \tau(t) = 1 \quad \text{if } \delta = 0.$$

For showing convergence result regarding $\mathbf{z}(t)$ only, we always assume that it is solution of (3.5) and for $(\mathbf{z}(t), q(t))$ its a solution of (3.4).

Lemma 3.1. *Suppose (A1), $\limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\mathbf{F}(t)\|^2 \leq M$ and $\mathbf{z}_0 \in \mathbf{H}_0^1$ hold. Then,*

$$\begin{aligned} &\tau^\beta(t) (\|e^{\alpha_1 t} \mathbf{z}\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2) + \gamma_1 e^{-2\delta_0 t} \int_0^t \tau^\beta(s) \|e^{\alpha s} \nabla \mathbf{z}(s)\|^2 ds \\ &\leq e^{-2\delta_0 t} \tau^\beta(0) (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2) + \frac{2}{\lambda_1 \gamma_1} e^{-2\delta_0 t} \int_0^t \tau^\beta(s) \|e^{\alpha s} \mathbf{F}\|^2 ds, \end{aligned}$$

where $\lambda_1 > 0$ is the minimum eigenvalue of the Dirichlet eigenvalue problem for the Laplace operator. Moreover,

$$(3.7) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) (\|e^{\alpha_1 t} \mathbf{z}\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2) \leq \frac{M}{\lambda_1 \gamma_1 \delta_0},$$

and

$$(3.8) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\nabla \mathbf{z}(t)\|^2 \leq \frac{M}{\lambda_1 \gamma_1^2 \delta_0}.$$

Proof. Set $\phi = e^{2\alpha t} \mathbf{z}$ in (3.5) to obtain

$$(3.9) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|e^{\alpha t} \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}\|^2) - \alpha (\|e^{\alpha t} \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}\|^2) \\ &+ e^{2\alpha t} (\nu a(\mathbf{z}, \mathbf{z}) + b(\mathbf{z}, \mathbf{z}, \mathbf{z}) + b(\mathbf{u}^\infty, \mathbf{z}, \mathbf{z}) + b(\mathbf{z}, \mathbf{u}^\infty, \mathbf{z})) = (e^{\alpha t} \mathbf{F}, e^{\alpha t} \mathbf{z}). \end{aligned}$$

Use (2.17), (3.6), the Poincarè inequality (2.1) and the Young's inequality to obtain

$$(3.10) \quad \begin{aligned} &\frac{d}{dt} (\|e^{\alpha t} \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}\|^2) + 2\gamma_1 \|e^{\alpha t} \nabla \mathbf{z}\|^2 \leq \frac{2\alpha}{\lambda_1} (1 + \kappa\lambda_1) \|e^{\alpha t} \nabla \mathbf{z}\|^2 + 2(e^{\alpha t} \mathbf{F}, e^{\alpha t} \mathbf{z}) \\ &\leq \frac{\gamma_1}{2} \|e^{\alpha t} \nabla \mathbf{z}\|^2 + \frac{2}{\lambda_1 \gamma_1} \|e^{\alpha t} \mathbf{F}\|^2 + \frac{\gamma_1}{2} \|e^{\alpha t} \nabla \mathbf{z}\|^2. \end{aligned}$$

Therefore, we arrive at

$$(3.11) \quad \frac{d}{dt} (\|e^{\alpha t} \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}\|^2) + \gamma_1 \|e^{\alpha t} \nabla \mathbf{z}\|^2 \leq \frac{2}{\lambda_1 \gamma_1} \|e^{\alpha t} \mathbf{F}\|^2.$$

Multiply (3.11) with $\tau^\beta(t)$ to obtain

$$(3.12) \quad \begin{aligned} &\frac{d}{dt} \left(\tau^\beta(t) (\|e^{\alpha t} \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}\|^2) \right) + \gamma_1 \tau^\beta(t) \|e^{\alpha t} \nabla \mathbf{z}\|^2 \\ &\leq \frac{2}{\lambda_1 \gamma_1} \tau^\beta(t) \|e^{\alpha t} \mathbf{F}\|^2 + \frac{d}{dt} (\tau^\beta(t) (\|e^{\alpha t} \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}\|^2)). \end{aligned}$$

Now for $\bar{t} \geq 2\beta \frac{(1+\kappa\lambda_1)}{\gamma_1\lambda_1}$, it follows that

$$(3.13) \quad \frac{d}{dt}\tau^\beta(t) = 0, \quad \forall t \geq 0 \quad \text{and} \quad \beta = 0; \quad \frac{d}{dt}\tau^\beta(t) = 0 \quad \forall 0 < t < \bar{t} \quad \text{and} \quad \beta > 0.$$

For $t > \bar{t} \geq 2\beta \frac{(1+\kappa\lambda_1)}{\gamma_1\lambda_1}$, apply Poincaré inequality to obtain

$$(3.14) \quad \begin{aligned} \frac{d}{dt}(\tau^\beta(t))(\|e^{\alpha t}\mathbf{z}\|^2 + \kappa\|e^{\alpha t}\nabla\mathbf{z}\|^2) &\leq \beta\tau^{\beta-1}(t)\left(\frac{1}{\lambda_1} + \kappa\right)\|e^{\alpha t}\nabla\mathbf{z}\|^2 \\ &\leq \frac{\gamma_1}{2}\bar{t}\tau^{\beta-1}(t)\|e^{\alpha t}\nabla\mathbf{z}\|^2 \leq \frac{\gamma_1}{2}\tau^\beta(t)\|e^{\alpha t}\nabla\mathbf{z}\|^2. \end{aligned}$$

Therefore, using (3.14) in (3.12), we arrive at

$$(3.15) \quad \frac{d}{dt}\left(\tau^\beta(t)(\|e^{\alpha t}\mathbf{z}\|^2 + \kappa\|e^{\alpha t}\nabla\mathbf{z}\|^2)\right) + \frac{\gamma_1}{2}\tau^\beta(t)\|e^{\alpha t}\nabla\mathbf{z}\|^2 \leq \frac{2}{\lambda_1\gamma_1}\tau^\beta(t)\|e^{\alpha t}\mathbf{F}\|^2.$$

Integrate (3.15) with respect to t from 0 to t and then, multiply the resulting inequality by $e^{-2\delta_0 t}$ to obtain

$$(3.16) \quad \begin{aligned} &\tau^\beta(t)(\|e^{\alpha_1 t}\mathbf{z}\|^2 + \kappa\|e^{\alpha_1 t}\nabla\mathbf{z}\|^2) + \frac{\gamma_1}{2}e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds \\ &\leq \tau^\beta(0)e^{-2\delta_0 t}(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2) + \frac{2}{\lambda_1\gamma_1}e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|e^{\alpha s}\mathbf{F}\|^2 ds. \end{aligned}$$

A use of L'Hospital's rule, yields

$$(3.17) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \tau^\beta(t)(\|e^{\alpha_1 t}\mathbf{z}\|^2 + \kappa\|e^{\alpha_1 t}\nabla\mathbf{z}\|^2) &\leq \frac{2}{\lambda_1\gamma_1} \limsup_{t \rightarrow \infty} e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|e^{\alpha s}\mathbf{F}\|^2 ds \\ &= \frac{1}{\lambda_1\gamma_1\delta_0} \limsup_{t \rightarrow \infty} \tau^\beta(t)e^{2\alpha_1 t}\|\mathbf{F}\|^2 \\ &\leq \frac{M}{\lambda_1\gamma_1\delta_0}, \end{aligned}$$

and

$$(3.18) \quad \limsup_{t \rightarrow \infty} \gamma_1 e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds \leq \frac{M}{\lambda_1\gamma_1\delta_0},$$

i.e.

$$(3.19) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t)\gamma_1 e^{2\alpha_1 t}\|\nabla\mathbf{z}(t)\|^2 \leq \frac{M}{\lambda_1\gamma_1\delta_0}.$$

This completes the proof. \square

Lemma 3.2. *Under the assumption (A1), let $\limsup_{t \rightarrow \infty} \tau^\beta(t)e^{2\alpha_1 t}\|\mathbf{F}(t)\|^2 \leq M$ and $\mathbf{z}_0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that*

$$\begin{aligned} &\tau^\beta(t)(\|e^{\alpha_1 t}\nabla\mathbf{z}\|^2 + \kappa\|e^{\alpha_1 t}\tilde{\Delta}\mathbf{z}\|^2) + \frac{\nu}{2}e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|e^{\alpha s}\tilde{\Delta}\mathbf{z}(s)\|^2 ds \\ &\leq Ce^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\mathbf{F}\|^2 ds + Ce^{-2\delta_0 t}\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2) \\ &\quad + Ce^{-2\delta_0 t} \int_0^t \|\mathbf{z}(s)\|^2 \|\nabla\mathbf{z}(s)\|^2 \|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds. \end{aligned}$$

Moreover,

$$(3.20) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) (\|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + \kappa \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}\|^2) \leq \frac{CM}{2\delta_0},$$

and

$$(3.21) \quad \limsup_{t \rightarrow \infty} e^{2\alpha_1 t} \tau^\beta(t) \|\tilde{\Delta} \mathbf{z}(t)\|^2 \leq \frac{CM}{\nu \delta_0}.$$

Proof. Set $\phi = -e^{2\alpha t} \tilde{\Delta} \mathbf{z}$ in (3.5) to obtain

$$(3.22) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|e^{\alpha t} \nabla \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) - \alpha (\|e^{\alpha t} \nabla \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) + \nu \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \\ &= (e^{\alpha t} \mathbf{F}, -e^{\alpha t} \tilde{\Delta} \mathbf{z}) + e^{2\alpha t} \left(b(\mathbf{z}, \mathbf{z}, \tilde{\Delta} \mathbf{z}) + b(\mathbf{u}^\infty, \mathbf{z}, \tilde{\Delta} \mathbf{z}) + b(\mathbf{z}, \mathbf{u}^\infty, \tilde{\Delta} \mathbf{z}) \right). \end{aligned}$$

The term $b(\mathbf{z}, \mathbf{z}, \tilde{\Delta} \mathbf{z})$ is bounded by

$$(3.23) \quad \begin{aligned} b(\mathbf{z}, \mathbf{z}, \tilde{\Delta} \mathbf{z}) &\leq N \|\mathbf{z}\|^{1/2} \|\nabla \mathbf{z}\| \|\tilde{\Delta} \mathbf{z}\|^{3/2} \\ &\leq \frac{\nu}{12} \|\tilde{\Delta} \mathbf{z}\|^2 + \frac{1}{4} \left(\frac{9}{\nu} \right)^3 \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^4. \end{aligned}$$

The term $b(\mathbf{u}^\infty, \mathbf{z}, \tilde{\Delta} \mathbf{z}) + b(\mathbf{z}, \mathbf{u}^\infty, \tilde{\Delta} \mathbf{z})$ is bounded by

$$(3.24) \quad \begin{aligned} b(\mathbf{u}^\infty, \mathbf{z}, \tilde{\Delta} \mathbf{z}) + b(\mathbf{z}, \mathbf{u}^\infty, \tilde{\Delta} \mathbf{z}) &\leq 2N \left(\frac{1}{\lambda_1} \right)^{\frac{1}{4}} \|\nabla \mathbf{u}^\infty\|^{1/2} \|\tilde{\Delta} \mathbf{u}^\infty\|^{1/2} \|\nabla \mathbf{z}\| \|\tilde{\Delta} \mathbf{z}\| \\ &\leq \frac{\nu}{12} \|\tilde{\Delta} \mathbf{z}\|^2 + \frac{12}{\nu \sqrt{\lambda_1}} N^2 \|\nabla \mathbf{u}^\infty\| \|\tilde{\Delta} \mathbf{u}^\infty\| \|\nabla \mathbf{z}\|^2. \end{aligned}$$

The term $(e^{\alpha t} \mathbf{F}, -e^{\alpha t} \tilde{\Delta} \mathbf{z})$ is bounded by

$$(3.25) \quad (e^{\alpha t} \mathbf{F}, -e^{\alpha t} \tilde{\Delta} \mathbf{z}) \leq \frac{\nu}{12} \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 + \frac{3}{\nu} \|e^{\alpha t} \mathbf{F}\|^2.$$

Therefore from (3.22) using (3.23), (3.24), (3.25) and Poincaré inequality, we arrive at

$$(3.26) \quad \begin{aligned} & \frac{d}{dt} (\|e^{\alpha t} \nabla \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) + 2\nu \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \\ & \leq 2\alpha \frac{(1 + \lambda_1 \kappa)}{\lambda_1} \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 + \frac{\nu}{2} \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 + \frac{6}{\nu} \|e^{\alpha t} \mathbf{F}\|^2 \\ & + \frac{24}{\nu \sqrt{\lambda_1}} N^2 \|\nabla \mathbf{u}^\infty\| \|\tilde{\Delta} \mathbf{u}^\infty\| \|e^{\alpha t} \nabla \mathbf{z}\|^2 + \frac{1}{2} \left(\frac{9}{\nu} \right)^3 \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2. \end{aligned}$$

Consequently using (3.6) it follows that

$$(3.27) \quad \begin{aligned} & \frac{d}{dt} (\|e^{\alpha t} \nabla \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) + \nu \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \\ & \leq \frac{6}{\nu} \|e^{\alpha t} \mathbf{F}\|^2 + \frac{24}{\nu \sqrt{\lambda_1}} N^2 \|\nabla \mathbf{u}^\infty\| \|\tilde{\Delta} \mathbf{u}^\infty\| \|e^{\alpha t} \nabla \mathbf{z}\|^2 + \frac{1}{2} \left(\frac{9}{\nu} \right)^3 \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2. \end{aligned}$$

Now multiplying by $\tau^\beta(t)$ to the equation (3.27), we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\tau^\beta(t) (\|e^{\alpha t} \nabla \mathbf{z}\|^2 + \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) \right) + \nu \tau^\beta(t) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \\ & \leq \frac{6}{\nu} \tau^\beta(t) \|e^{\alpha t} \mathbf{F}\|^2 + \frac{24}{\nu \sqrt{\lambda_1}} N^2 \|\nabla \mathbf{u}^\infty\| \|\tilde{\Delta} \mathbf{u}^\infty\| \tau^\beta(t) \|e^{\alpha t} \nabla \mathbf{z}\|^2 \\ & + \frac{1}{2} \left(\frac{9}{\nu} \right)^3 \tau^\beta(t) \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2 + \frac{d}{dt} (\tau^\beta(t)) \left(\frac{1}{\lambda_1} + \kappa \right) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2. \end{aligned}$$

For $\bar{t} \geq \frac{2\beta(1+\kappa\lambda_1)}{\nu\lambda_1}$, it follows that

$$\frac{d}{dt}(\tau^\beta(t))\left(\frac{1}{\lambda_1} + \kappa\right)\|e^{\alpha t}\tilde{\Delta}\mathbf{z}\|^2 = \beta\tau^{\beta-1}(t)\frac{(1+\kappa\lambda_1)}{\lambda_1}\|e^{\alpha t}\tilde{\Delta}\mathbf{z}\|^2 \leq \frac{\nu}{2}\bar{t}\tau^{\beta-1}(t)\|e^{\alpha t}\tilde{\Delta}\mathbf{z}\|^2 \leq \frac{\nu}{2}\tau^\beta(t)\|e^{\alpha t}\tilde{\Delta}\mathbf{z}\|^2.$$

Integrate from 0 to t and then multiply the resulting inequality by $e^{-2\delta_0 t}$. A use of Lemma 3.1 leads to

$$\begin{aligned} & \tau^\beta(t)(\|e^{\alpha_1 t}\nabla\mathbf{z}\|^2 + \kappa\|e^{\alpha_1 t}\tilde{\Delta}\mathbf{z}\|^2) + \frac{\nu}{2}e^{-2\delta_0 t}\int_0^t \tau^\beta(s)\|e^{\alpha s}\tilde{\Delta}\mathbf{z}\|^2 \\ & \leq Ce^{-2\delta_0 t}\int_0^t \tau^\beta(s)\|e^{\alpha s}\mathbf{F}\|^2 ds + Ce^{-2\delta_0 t}\tau^\beta(0)(\|\mathbf{z}_0\|^2 + (1\kappa)\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2) \\ (3.28) \quad & + Ce^{-2\delta_0 t}\int_0^t \|\mathbf{z}(s)\|^2\|\nabla\mathbf{z}(s)\|^2\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds. \end{aligned}$$

Now apply L'Hospital's rule and put $\alpha_1 = 0$ in (3.7) and (3.8) to obtain

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \tau^\beta(t)(\|e^{\alpha_1 t}\nabla\mathbf{z}\|^2 + \kappa\|e^{\alpha_1 t}\tilde{\Delta}\mathbf{z}\|^2) \\ & \leq C\frac{1}{2\delta_0}(\limsup_{t \rightarrow \infty} \tau^\beta(t)\|e^{\alpha_1 t}\mathbf{F}\|^2 + \limsup_{t \rightarrow \infty} \tau^\beta(t)\|\mathbf{z}(t)\|^2\|\nabla\mathbf{z}(t)\|^2\|e^{\alpha_1 t}\nabla\mathbf{z}(t)\|^2) \\ & \leq C\frac{1}{2\delta_0}\limsup_{t \rightarrow \infty} \tau^\beta(t)\|e^{\alpha_1 t}\mathbf{F}\|^2, \end{aligned}$$

and

$$\limsup_{t \rightarrow \infty} \frac{\nu}{2\delta_0}\|e^{\alpha_1 t}\tilde{\Delta}\mathbf{z}(t)\|^2 \leq C\frac{1}{2\delta_0}\limsup_{t \rightarrow \infty} \tau^\beta(t)\|e^{\alpha_1 t}\mathbf{F}\|^2.$$

This concludes the proof. \square

Lemma 3.3. *Under the assumption (A1), let $\limsup_{t \rightarrow \infty} \tau^\beta(t)e^{2\alpha_1 t}\|\mathbf{F}(t)\|^2 \leq M$ and $\mathbf{z}_0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that*

$$\begin{aligned} & \tau^\beta(t)\|e^{\alpha_1 t}\nabla\mathbf{z}\|^2 + e^{-2\delta_0 t}\int_0^t \tau^\beta(s)e^{2\alpha s}(\|\mathbf{z}_t(s)\|^2 + 2\kappa\|\nabla\mathbf{z}_t(s)\|^2)ds \\ & \leq Ce^{-2\delta_0 t}\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2) \\ & + Ce^{-2\delta_0 t}\int_0^t \tau^\beta(s)\|e^{\alpha s}\mathbf{F}\|^2 ds + Ce^{-2\delta_0 t}\int_0^t \tau^\beta(s)\|\mathbf{z}(s)\|^2\|\nabla\mathbf{z}(s)\|^2\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds. \end{aligned}$$

Moreover,

$$(3.29) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t)e^{2\alpha_1 t}(\|\nabla\mathbf{z}\|^2) \leq CM\frac{1}{2\delta_0},$$

and

$$(3.30) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t)e^{2\alpha_1 t}(\|\mathbf{z}_t(t)\|^2 + 2\kappa\|\nabla\mathbf{z}_t(t)\|^2) \leq CM.$$

Proof. Set $\phi = e^{2\alpha t}\mathbf{z}_t$ in (3.5) to obtain

$$\begin{aligned} & \|e^{\alpha t}\mathbf{z}_t\|^2 + \kappa\|e^{\alpha t}\nabla\mathbf{z}_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\|e^{\alpha t}\nabla\mathbf{z}\|^2 = \nu\alpha\|e^{\alpha t}\nabla\mathbf{z}\|^2 + (e^{\alpha t}\mathbf{F}, e^{\alpha t}\mathbf{z}_t) - e^{\alpha t}b(\mathbf{z}, \mathbf{z}, e^{\alpha t}\mathbf{z}_t) \\ (3.31) \quad & - b(\mathbf{u}^\infty, e^{\alpha t}\mathbf{z}, e^{\alpha t}\mathbf{z}_t) - b(e^{\alpha t}\mathbf{z}, \mathbf{u}^\infty, e^{\alpha t}\mathbf{z}_t). \end{aligned}$$

The term $(e^{\alpha t} \mathbf{F}, e^{\alpha t} \mathbf{z}_t)$ is estimated as

$$(e^{\alpha t} \mathbf{F}, e^{\alpha t} \mathbf{z}_t) \leq \frac{3}{2} \|e^{\alpha t} \mathbf{F}\|^2 + \frac{1}{6} \|e^{\alpha t} \mathbf{z}_t\|^2.$$

The term $e^{\alpha t} b(\mathbf{z}, \mathbf{z}, e^{\alpha t} \mathbf{z}_t)$ is bounded by

$$\begin{aligned} e^{\alpha t} b(\mathbf{z}, \mathbf{z}, e^{\alpha t} \mathbf{z}_t) &\leq e^{\alpha t} \|\mathbf{z}\|_{L^4} \|\nabla \mathbf{z}\|_{L^4} \|e^{\alpha t} \mathbf{z}_t\| \\ &\leq C e^{\alpha t} \|\mathbf{z}\|^{1/2} \|\nabla \mathbf{z}\| \|\tilde{\Delta} \mathbf{z}\|^{1/2} \|e^{\alpha t} \mathbf{z}_t\| \\ &\leq \frac{1}{6} \|e^{\alpha t} \mathbf{z}_t\|^2 + \frac{3C^2}{2} \|\mathbf{z}\| \|\nabla \mathbf{z}\| \|e^{\alpha t} \nabla \mathbf{z}\| \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\| \\ &\leq \frac{1}{6} \|e^{\alpha t} \mathbf{z}_t\|^2 + C \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2 + C \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2. \end{aligned}$$

For the term $b(\mathbf{u}^\infty, e^{\alpha t} \mathbf{z}, e^{\alpha t} \mathbf{z}_t) + b(e^{\alpha t} \mathbf{z}, \mathbf{u}^\infty, e^{\alpha t} \mathbf{z}_t)$, we note that

$$\begin{aligned} b(\mathbf{u}^\infty, e^{\alpha t} \mathbf{z}, e^{\alpha t} \mathbf{z}_t) + b(e^{\alpha t} \mathbf{z}, \mathbf{u}^\infty, e^{\alpha t} \mathbf{z}_t) &\leq 2N \frac{1}{\lambda_1^4} \|\nabla \mathbf{u}^\infty\|^{1/2} \|\tilde{\Delta} \mathbf{u}^\infty\|^{1/2} \|e^{\alpha t} \nabla \mathbf{z}\| \|e^{\alpha t} \mathbf{z}_t\| \\ &\leq \frac{1}{6} \|e^{\alpha t} \mathbf{z}_t\|^2 + 6N^2 \frac{1}{\lambda_1^2} \|\nabla \mathbf{u}^\infty\| \|\tilde{\Delta} \mathbf{u}^\infty\| \|e^{\alpha t} \nabla \mathbf{z}\|^2. \end{aligned}$$

On substitution, we arrive at from (3.31) that

$$\begin{aligned} &\|e^{\alpha t} \mathbf{z}_t\|^2 + 2\kappa \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 + \nu \frac{d}{dt} \|e^{\alpha t} \nabla \mathbf{z}\|^2 \\ &\leq 3 \|e^{\alpha t} \mathbf{F}\|^2 + C (\|e^{\alpha t} \nabla \mathbf{z}\|^2 + \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) + C \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2. \end{aligned}$$

Multiply the above inequality by $\tau^\beta(t)$ to obtain

$$\begin{aligned} \nu \frac{d}{dt} (\tau^\beta(t) \|e^{\alpha t} \nabla \mathbf{z}\|^2) + \tau^\beta(t) (\|e^{\alpha t} \mathbf{z}_t\|^2 + 2\kappa \|e^{\alpha t} \nabla \mathbf{z}_t\|^2) &\leq 3\tau^\beta(t) \|e^{\alpha t} \mathbf{F}\|^2 + C\tau^\beta(t) (\|e^{\alpha t} \nabla \mathbf{z}\|^2 + \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) \\ &\quad + C\tau^\beta(t) \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2. \end{aligned}$$

Integrate above inequality from 0 to t and then, multiply the resulting inequality by $e^{-2\delta_0 t}$ to obtain

$$\begin{aligned} &\tau^\beta(t) \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + e^{-2\delta_0 t} \int_0^t e^{2\alpha s} \tau^\beta(s) (\|\mathbf{z}_t(s)\|^2 + 2\kappa \|\nabla \mathbf{z}_t(s)\|^2) ds \\ &\leq e^{-2\delta_0 t} \tau^\beta(0) \|\nabla \mathbf{z}_0\|^2 + C e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} (\|\nabla \mathbf{z}(s)\|^2 + \|\tilde{\Delta} \mathbf{z}(s)\|^2) ds \\ (3.32) \quad &+ C e^{-2\delta_0 t} \int_0^t \tau^\beta(s) \|\mathbf{z}(s)\|^2 \|\nabla \mathbf{z}(s)\|^2 \|e^{\alpha t} \nabla \mathbf{z}(s)\|^2 ds. \end{aligned}$$

An application of Lemmas 3.1, 3.2 in (3.32) implies that

$$\begin{aligned} &\tau^\beta(t) \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + e^{-2\delta_0 t} \int_0^t e^{2\alpha s} (\|\mathbf{z}_t(s)\|^2 + 2\kappa \|\nabla \mathbf{z}_t(s)\|^2) ds \\ &\leq C e^{-2\delta_0 t} (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C e^{-2\delta_0 t} \int_0^t \tau^\beta(s) \|e^{\alpha s} \mathbf{F}\|^2 ds \\ (3.33) \quad &+ C e^{-2\delta_0 t} \int_0^t \tau^\beta(s) \|\mathbf{z}(s)\|^2 \|\nabla \mathbf{z}(s)\|^2 \|e^{\alpha t} \nabla \mathbf{z}(s)\|^2 ds. \end{aligned}$$

Now as $t \rightarrow \infty$, using L'Hospital's rule, we obtain from (3.33) as in Lemmas 3.1, 3.2

$$\limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} (\|\nabla \mathbf{z}\|^2) \leq CM \frac{1}{2\delta_0},$$

$$\limsup_{t \rightarrow \infty} e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} (\|\mathbf{z}_t(s)\|^2 + 2\kappa \|\nabla \mathbf{z}_t(s)\|^2) ds \leq CM \frac{1}{2\delta_0}.$$

Hence,

$$\limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} (\|\mathbf{z}_t(t)\|^2 + 2\kappa \|\nabla \mathbf{z}_t(t)\|^2) \leq CM.$$

This completes the rest of the proof. \square

Lemma 3.4. *Under the assumption (A1), let $\tau^\beta(t) e^{2\alpha_1 t} (\|\mathbf{F}(t)\|^2 + \|\mathbf{F}_t\|_{-1}^2) \leq M_1$ and $\mathbf{z}_0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that*

$$\begin{aligned} & \tau^\beta(t) (\|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}_t\|^2) + \nu e^{-2\delta_0 t} \int_0^t e^{2\alpha s} \|\nabla \mathbf{z}_t(s)\|^2 ds \\ & \leq C e^{-2\delta_0 t} \tau^\beta(0) (\|\mathbf{F}_0\|^2 + \|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) \\ & \quad + C(M_1) \frac{(1 - e^{-2\delta_0 t})}{2\delta_0}. \end{aligned}$$

Moreover,

$$(3.34) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} (\|\mathbf{z}_t\|^2 + \kappa \|\nabla \mathbf{z}_t\|^2) \leq C(M_1) \frac{1}{2\delta_0},$$

and

$$(3.35) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\nabla \mathbf{z}_t(t)\|^2 \leq C \frac{(M_1)}{\nu}.$$

Proof. Differentiate (3.5) with respect to t and set $\phi = e^{2\alpha t} \mathbf{z}_t$ to obtain

$$\begin{aligned} & (\mathbf{z}_{tt}, e^{2\alpha t} \mathbf{z}_t) + \kappa (\nabla \mathbf{z}_{tt}, e^{2\alpha t} \nabla \mathbf{z}_t) + \nu (\nabla \mathbf{z}_t, e^{2\alpha t} \nabla \mathbf{z}_t) \\ (3.36) \quad & = ((\mathbf{z} \cdot \nabla \mathbf{z})_t, -e^{2\alpha t} \mathbf{z}_t) - e^{2\alpha t} b(\mathbf{u}^\infty, \mathbf{z}_t, \mathbf{z}_t) - e^{2\alpha t} b(\mathbf{z}_t, \mathbf{u}^\infty, \mathbf{z}_t) + (\mathbf{F}_t, e^{2\alpha t} \mathbf{z}_t). \end{aligned}$$

Now (3.36) can be written as

$$\begin{aligned} & \frac{d}{dt} (\|e^{\alpha t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}_t\|^2) + 2\nu \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 \\ (3.37) \quad & = 2\alpha (\|e^{\alpha t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}_t\|^2) + 2(e^{\alpha t} \mathbf{F}_t, e^{\alpha t} \mathbf{z}_t) - 2b(e^{\alpha t} \mathbf{z}_t, \mathbf{z}, e^{\alpha t} \mathbf{z}_t) - 2b(\mathbf{z}, e^{\alpha t} \mathbf{z}_t, e^{\alpha t} \mathbf{z}_t) \\ & \quad - 2(\mathbf{u}^\infty, e^{\alpha t} \mathbf{z}_t, e^{\alpha t} \mathbf{z}_t) - 2b(e^{\alpha t} \mathbf{z}_t, \mathbf{u}^\infty, e^{\alpha t} \mathbf{z}_t). \end{aligned}$$

The right hand side terms of (3.37) are bounded by

$$2\alpha (\|e^{\alpha t} \mathbf{z}_t\|^2 + \|e^{\alpha t} \nabla \mathbf{z}_t\|^2) \leq \frac{2\alpha(1 + \kappa\lambda_1)}{\lambda_1} \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 \leq \frac{\nu}{2} \|e^{\alpha t} \nabla \mathbf{z}_t\|^2,$$

$$2(e^{\alpha t} \mathbf{F}_t, e^{\alpha t} \mathbf{z}_t) \leq \frac{\nu}{6} \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 + \frac{6}{\nu} \|e^{\alpha t} \mathbf{F}_t\|_{-1}^2,$$

$$\begin{aligned} 2b(e^{\alpha t} \mathbf{z}_t, \mathbf{z}, e^{\alpha t} \mathbf{z}_t) + 2b(\mathbf{z} \cdot e^{\alpha t} \nabla \mathbf{z}_t, e^{\alpha t} \mathbf{z}_t) & = 2b(e^{\alpha t} \mathbf{z}_t \cdot \nabla \mathbf{z}, e^{\alpha t} \mathbf{z}_t) \leq C \|\nabla \mathbf{z}\| \|e^{\alpha t} \mathbf{z}_t\| \|e^{\alpha t} \nabla \mathbf{z}_t\| \\ & \leq \frac{\nu}{6} \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 + C(\nu) \|e^{\alpha t} \mathbf{z}_t\|^2 \|\nabla \mathbf{z}\|^2, \end{aligned}$$

and

$$\begin{aligned} 2b(\mathbf{u}^\infty, e^{\alpha t} \mathbf{z}_t, e^{\alpha t} \mathbf{z}_t) + b(e^{\alpha t} \mathbf{z}_t, \mathbf{u}^\infty, e^{\alpha t} \mathbf{z}_t) & = 2b(e^{\alpha t} \mathbf{z}_t, \mathbf{u}^\infty, e^{\alpha t} \mathbf{z}_t) \leq C \|e^{\alpha t} \mathbf{z}_t\| \|\nabla \mathbf{u}^\infty\| \|e^{\alpha t} \nabla \mathbf{z}_t\| \\ & \leq \frac{\nu}{6} \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 + C \|e^{\alpha t} \mathbf{z}_t\|^2. \end{aligned}$$

Hence we arrive at from (3.37)

$$\frac{d}{dt}(\|e^{\alpha t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}_t\|^2) + \nu \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 \leq \frac{6}{\nu} \|e^{\alpha t} \mathbf{F}_t\|_{-1}^2 + C \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \mathbf{z}_t\|^2.$$

Multiply the above inequality by $\tau^\beta(t)$ to obtain

$$\begin{aligned} \frac{d}{dt} \left(\tau^\beta(t) (\|e^{\alpha t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha t} \nabla \mathbf{z}_t\|^2) \right) + \nu \tau^\beta(t) \|e^{\alpha t} \nabla \mathbf{z}_t\|^2 \\ \leq \frac{6}{\nu} \tau^\beta(t) \|e^{\alpha t} \mathbf{F}_t\|_{-1}^2 + C \tau^\beta(t) \|\nabla \mathbf{z}(t)\|^2 \|e^{\alpha t} \mathbf{z}_t\|^2. \end{aligned}$$

Integrate from 0 to t and then multiply the resulting inequality by $e^{-2\delta_0 t}$ to obtain

$$\begin{aligned} \tau^\beta(t) (\|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}_t\|^2) + \nu e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} \|\nabla \mathbf{z}_t(s)\|^2 ds \\ \leq e^{-2\delta_0 t} \tau^\beta(0) (\|\mathbf{z}_t(0)\|^2 + \|\nabla \mathbf{z}_t(0)\|^2) + \frac{6}{\nu} e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} \|\mathbf{F}_t\|_{-1}^2 ds \\ + C e^{-2\delta_0 t} \int_0^t e^{2\alpha s} \|\nabla \mathbf{z}(s)\|^2 \|\mathbf{z}_t(s)\|^2 ds. \end{aligned} \quad (3.38)$$

Now from equation (3.5) after putting $\phi = \mathbf{z}_t$, it follows that

$$(3.39) \quad \|\mathbf{z}_t\|^2 + \kappa \|\nabla \mathbf{z}_t\|^2 \leq C(\nu, \lambda_1, \|f^\infty\|_{-1}) (\|\mathbf{F}\|^2 + \|\tilde{\Delta} \mathbf{z}\|^2 + \|\nabla \mathbf{z}\|^2 \|\tilde{\Delta} \mathbf{z}\|^2).$$

From (3.39), we obtain the estimate at $t = 0$ i.e. $\|\mathbf{z}_t(0)\|^2 + \kappa \|\nabla \mathbf{z}_t(0)\|^2$. Use previous Lemma 3.3 in (3.38) to obtain

$$\begin{aligned} \tau^\beta(t) (\|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}_t\|^2) + \nu e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} \|\nabla \mathbf{z}_t(s)\|^2 ds \\ \leq C e^{-2\delta_0 t} \tau^\beta(0) (\|\mathbf{F}_0\|^2 + \|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) \\ + C e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} (\|F(s)\|^2 + \|F_t(s)\|_{-1}^2) ds \\ \leq C e^{-2\delta_0 t} \tau^\beta(0) (\|\mathbf{F}_0\|^2 + \|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) \\ + C(M_1) \frac{(1 - e^{-2\delta_0 t})}{2\delta_0}. \end{aligned} \quad (3.40)$$

This completes the first part of the proof. Use L'Hospital's rule to obtain from (3.40)

$$\limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} (\|\mathbf{z}_t\|^2 + \kappa \|\nabla \mathbf{z}_t\|^2) \leq C(M_1) \frac{1}{2\delta_0},$$

and

$$\limsup_{t \rightarrow \infty} \nu e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} \|\nabla \mathbf{z}_t(s)\|^2 ds \leq C(M_1) \frac{1}{2\delta_0},$$

and hence,

$$\limsup_{t \rightarrow \infty} \nu \tau^\beta(t) e^{2\alpha_1 t} \|\nabla \mathbf{z}_t(t)\|^2 \leq C(M_1).$$

This completes the rest of the proof. □

Lemma 3.5. *Under the assumption (A1), let $\tau^\beta(t) e^{2\alpha_1 t} \|\mathbf{F}(t)\|^2 \leq M \forall t \geq 0$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that*

$$(3.41) \quad \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{z}(t)\|_{\mathbf{H}^1}^2 \leq CM \quad \forall t, \beta \geq 0,$$

holds.

Proof. From Lemma 3.1, it follows that

$$\begin{aligned}
\tau^\beta(t)(\|e^{\alpha_1 t} \mathbf{z}(t)\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}(t)\|^2) &\leq \tau^\beta(0)e^{-2\delta_0 t}(\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2) + \frac{2}{\gamma_1 \lambda_1} e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha_1 s} \|\mathbf{F}\|^2 ds \\
&\leq \tau^\beta(0)e^{-2\delta_0 t}(\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2) + \frac{2}{\gamma_1 \lambda_1} M \frac{1 - e^{-2\delta_0 t}}{2\delta_0} \\
(3.42) \quad &\leq \tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2) + \frac{1}{\gamma_1 \lambda_1 \delta_0} M.
\end{aligned}$$

Also from Lemma 3.1, we find that

$$(3.43) \quad \frac{d}{dt}(\|e^{\alpha_1 t} \mathbf{z}(t)\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}(t)\|^2) + \frac{\gamma_1}{2} \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 \leq \frac{2}{\gamma_1 \lambda_1} \|e^{\alpha_1 t} \mathbf{F}\|^2.$$

Integrate (3.43) from 0 to t to obtain

$$\begin{aligned}
\frac{\gamma_1}{2} \int_0^t e^{2\alpha_1 s} \|\nabla \mathbf{z}(s)\|^2 ds &\leq (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2) + \frac{2}{\gamma_1 \lambda_1} \int_0^t e^{2\alpha_1 s} \|\mathbf{F}\|^2 ds \\
&\leq (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2) + \frac{2}{\gamma_1 \lambda_1} M t.
\end{aligned}$$

Therefore, we arrive at

$$(3.44) \quad \int_0^t \|\mathbf{z}(s)\|^2 \|\nabla \mathbf{z}(s)\|^2 ds \leq C(1+t) \quad \forall t \geq 0.$$

Now from (3.28) in Lemma 3.2, it follows that by using Gronwall's inequality

$$\begin{aligned}
\tau^\beta(t)(\|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + \kappa \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}\|^2) &\leq \left(e^{-2\delta_0 t} \tau^\beta(0) \{ \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2 \} + C(\nu) e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha_1 s} \|\mathbf{F}\|^2 ds \right) \\
&\times \exp \{ C(N, \nu, \lambda_1, \|f^\infty\|_{-1}) \int_0^t \|\mathbf{z}(s)\|^2 \|\nabla \mathbf{z}(s)\|^2 ds \} \\
&\leq \left(e^{-2\delta_0 t} \{ \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2 \} + C(\nu) M \frac{(1 - e^{-2\delta_0 t})}{2\delta_0} \right) \\
(3.45) \quad &\times \exp \{ C(M)(1+t) \}.
\end{aligned}$$

Now, (3.45) holds for finite t , $0 < t \leq T$, where $0 < T < \infty$. For large $t > T$, we note from Lemma 3.1, that

$$(3.46) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 \leq C \limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{F}\|^2 \leq CM.$$

Therefore from (3.46) and for a finite time $T > 0$ there holds

$$(3.47) \quad \tau^\beta(t) \|e^{\alpha_1 t} \nabla \mathbf{z}(t)\|^2 \leq CM \quad \forall t \geq T.$$

Hence, (3.47) with (3.45) and (3.42) complete the proof. \square

Lemma 3.6. *Under the assumption (A1), let $\tau^\beta(t) e^{2\alpha_1 t} (\|\mathbf{F}(t)\|^2 + \|\mathbf{F}_t\|_{-1}^2) \leq M_1 \quad \forall t \geq 0$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that for $t \geq 0$*

$$\tau^\beta(t) (\|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}_t\|^2) \leq C \tau^\beta(0) (\|\mathbf{F}_0\|^2 + \|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C \frac{(M_1)}{2\delta_0},$$

$$\nu \tau^\beta(t) \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}\|^2 \leq C \tau^\beta(0) (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M_1),$$

and

$$\tau^\beta(t) e^{2\alpha_1 t} \|\kappa \tilde{\Delta} \mathbf{z}_t\|^2 \leq C \tau^\beta(0) (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M_1),$$

hold.

Proof. From Lemma 3.4, we obtain

$$\begin{aligned}
& \tau^\beta(t)(\|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \kappa \|e^{\alpha_1 t} \nabla \mathbf{z}_t\|^2) + \nu e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} \|\nabla \mathbf{z}_t(s)\|^2 ds \\
& \leq C e^{-2\delta_0 t} \tau^\beta(0)(\|\mathbf{F}_0\|^2 + \|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) \\
& \quad + C M_1 \frac{(1 - e^{-2\delta_0 t})}{2\delta_0} \\
(3.48) \quad & \leq C e^{-2\delta_0 t} \tau^\beta(0)(\|\mathbf{F}_0\|^2 + \|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C \frac{(M_1)}{2\delta_0}.
\end{aligned}$$

From equation (3.5), it follows that

$$(3.49) \quad (\mathbf{z}_t, \phi) - \kappa(\tilde{\Delta} \mathbf{z}_t, \phi) - \nu(\tilde{\Delta} \mathbf{z}, \phi) + b(\mathbf{z}, \mathbf{z}, \phi) + b(\mathbf{u}^\infty, \mathbf{z}, \phi) + b(\mathbf{z}, \mathbf{u}^\infty, \phi) = (\mathbf{F}, \phi).$$

Now set $\phi = -e^{2\alpha t} \tilde{\Delta} \mathbf{z}$ in (3.49) to obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) - \alpha \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 + \nu \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \\
(3.50) \quad & = (e^{\alpha t} \mathbf{F}, -e^{\alpha t} \tilde{\Delta} \mathbf{z}) + e^{2\alpha t} b(\mathbf{z}, \mathbf{z}, \tilde{\Delta} \mathbf{z}) + b(\mathbf{u}^\infty, \mathbf{z}, \tilde{\Delta} \mathbf{z}) + b(\mathbf{z}, \mathbf{u}^\infty, \tilde{\Delta} \mathbf{z}) + (e^{\alpha t} \mathbf{z}_t, e^{\alpha t} \tilde{\Delta} \mathbf{z}).
\end{aligned}$$

From (3.50), it follows that after multiplying the above equation by $\tau^\beta(t)$ and bounding the right hand side term as in Lemma 3.2

$$\begin{aligned}
& \nu \tau^\beta(t) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 + \frac{d}{dt} (\kappa \tau^\beta(t) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) \\
& \leq 2\alpha \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 + \frac{6}{\nu} \|e^{\alpha t} \mathbf{F}\|^2 + \frac{24}{\nu \sqrt{\lambda_1}} N^2 \tau^\beta(t) \|\nabla \mathbf{u}^\infty\| \|\tilde{\Delta} \mathbf{u}^\infty\| \|e^{\alpha t} \nabla \mathbf{z}\|^2 \\
& \quad + \frac{1}{2} \left(\frac{9}{\nu}\right)^3 \tau^\beta(t) \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2 + \kappa \frac{d}{dt} (\tau^\beta(t) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2) + \frac{2}{\nu} \tau^\beta(t) \|e^{\alpha t} \mathbf{z}_t\|^2.
\end{aligned}$$

Again use Lemma 3.2 to find that

$$\begin{aligned}
& \nu \tau^\beta(t) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \leq C \tau^\beta(t) \kappa \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 + C \nu \tau^\beta(t) \|e^{\alpha t} \mathbf{F}\|^2 \\
& \quad + C \tau^\beta(t) \|e^{\alpha t} \nabla \mathbf{z}\|^2 + C \tau^\beta(t) \|e^{\alpha t} \mathbf{z}_t\|^2 + C \tau^\beta(t) \|\mathbf{z}(t)\|^2 \|\nabla \mathbf{z}(t)\|^2 \|e^{\alpha t} \nabla \mathbf{z}\|^2.
\end{aligned}$$

Therefore multiply by $e^{-2\delta_0 t}$ and use Lemma 3.2 to obtain

$$\begin{aligned}
& \nu \tau^\beta(t) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \leq C \nu \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{F}\|^2 + C \tau^\beta(t) \|\mathbf{z}(t)\|^2 \|\nabla \mathbf{z}(t)\|^2 \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + C \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{z}_t\|^2 \\
& \quad + C \tau^\beta(0) e^{-2\delta_0 t} (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C e^{-2\delta_0 t} \int_0^t \tau^\beta(s) e^{2\alpha s} \|\mathbf{F}\|^2 ds \\
(3.51) \quad & \quad + C e^{-2\delta_0 t} \int_0^t \tau^\beta(s) \|\mathbf{z}(s)\|^2 \|\nabla \mathbf{z}(s)\|^2 \|e^{\alpha s} \nabla \mathbf{z}(s)\|^2 ds.
\end{aligned}$$

A use of previous Lemma 3.5 with (3.48) implies that

$$(3.52) \quad \nu \tau^\beta(t) \|e^{\alpha t} \tilde{\Delta} \mathbf{z}\|^2 \leq C \tau^\beta(0) e^{-2\delta_0 t} (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M + M_1).$$

Now

$$\kappa \|\tilde{\Delta} \mathbf{z}_t\| \leq \|\mathbf{z}_t\| + \nu \|\tilde{\Delta} \mathbf{z}\| + \|\mathbf{z}\| \|\nabla \mathbf{z}\| + \|\mathbf{u}^\infty\| \|\nabla \mathbf{z}\| + \|\mathbf{z}\| \|\nabla \mathbf{u}^\infty\| + \|\mathbf{F}\|.$$

Therefore, we arrive at

$$\begin{aligned}
& \tau^\beta(t) e^{2\alpha_1 t} \|\kappa \tilde{\Delta} \mathbf{z}_t\|^2 \leq C \tau^\beta(t) e^{2\alpha_1 t} \|\mathbf{z}_t\|^2 + C \tau^\beta(t) e^{2\alpha_1 t} \|\tilde{\Delta} \mathbf{z}\|^2 + C \tau^\beta(t) \|\mathbf{z}\|^2 \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 \\
& \quad + C \tau^\beta(t) e^{2\alpha_1 t} \|\nabla \mathbf{z}\|^2 + C \tau^\beta(t) e^{2\alpha_1 t} \|\mathbf{F}\|^2.
\end{aligned}$$

A use of previous Lemmas and equation (3.52) yields

$$(3.53) \quad \tau^\beta(t) e^{2\alpha_1 t} \|\kappa \tilde{\Delta} \mathbf{z}_t\|^2 \leq C \tau^\beta(0) e^{-2\delta_0 t} (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M_1).$$

This completes the rest of the proof. \square

Lemma 3.7. *Under the assumption (A1), let $\tau^\beta(t) e^{2\alpha_1 t} (\|\mathbf{F}(t)\|^2 + \|\mathbf{F}_t\|_{-1}^2) \leq M_1 \forall t \geq 0$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that*

$$(3.54) \quad \tau^\beta(t) e^{2\alpha_1 t} \|q(t)\|_{H^1(\Omega)/\mathbb{R}}^2 \leq C \tau^\beta(0) e^{-2\delta_0 t} (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M_1).$$

Proof. Use property of the divergence free space \mathbf{J}_1 to obtain for $\phi \in \mathbf{H}_0^1$

$$(\nabla q, \phi) = (\mathbf{z}_t - \kappa \tilde{\Delta} \mathbf{z}_t - \nu \tilde{\Delta} \mathbf{z} + \mathbf{z} \cdot \nabla \mathbf{z} + \mathbf{u}^\infty \cdot \nabla \mathbf{z} + \mathbf{z} \cdot \nabla \mathbf{u}^\infty - \mathbf{F}, \phi).$$

Hence, it follows that

$$\begin{aligned} |(\nabla q, \phi)| &\leq \|\mathbf{z}_t\| \|\phi\| + \kappa \|\tilde{\Delta} \mathbf{z}_t\| \|\phi\| + \nu \|\tilde{\Delta} \mathbf{z}\| \|\phi\| + C \|\mathbf{z}\|^{1/2} \|\nabla \mathbf{z}\| \|\tilde{\Delta} \mathbf{z}\|^{1/2} \|\phi\| \\ &\quad + C(\nu, \lambda_1, \|f^\infty\|_{-1}) \|\nabla \mathbf{z}\| \|\phi\| + \|\mathbf{F}\| \|\phi\|, \end{aligned}$$

and

$$(3.55) \quad \begin{aligned} \tau^\delta(t) \|e^{\alpha_1 t} \nabla q\| &\leq \frac{|(\tau^\beta(t) e^{\alpha_1 t} \nabla q, \phi)|}{\|\phi\|} \leq C \tau^\delta(t) \left(\|e^{\alpha_1 t} \mathbf{z}_t\| + \kappa \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}_t\| + \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}\| \right. \\ &\quad \left. + e^{\alpha_1 t} \|\mathbf{z}\|^{1/2} \|\nabla \mathbf{z}\| \|\tilde{\Delta} \mathbf{z}\|^{1/2} + \|e^{\alpha_1 t} \nabla \mathbf{z}\| + \|e^{\alpha_1 t} \mathbf{F}\| \right). \end{aligned}$$

Hence, squaring both sides, we find that

$$\tau^\beta(t) \|e^{\alpha_1 t} \nabla q\|^2 \leq C \tau^\beta(t) e^{2\alpha_1 t} \left(\|\mathbf{z}_t\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_t\|^2 + \|\tilde{\Delta} \mathbf{z}\|^2 + \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 + \|\nabla \mathbf{z}\|^2 + \|\mathbf{F}\|^2 \right).$$

The above inequality can be rewritten as

$$(3.56) \quad \begin{aligned} \tau^\beta(t) \|e^{\alpha_1 t} \nabla q\|^2 &\leq C \left(\tau^\beta(t) \|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \tau^\beta(t) \|\kappa e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}_t\|^2 + \tau^\beta(t) \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}\|^2 \right. \\ &\quad \left. + \tau^\beta(t) \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + \tau^\beta(t) \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{F}\|^2 \right). \end{aligned}$$

Also from (2.3), it follows that

$$(3.57) \quad c \tau^\delta(t) \|e^{\alpha_1 t} q(t)\| \leq \sup_{v \in \mathbf{H}_0^1} \frac{(\mathbf{v}, \tau^\delta(t) e^{\alpha_1 t} \nabla q(t))}{\|\nabla \mathbf{v}\|} \leq \frac{1}{\sqrt{\lambda_1}} \tau^\delta(t) \|e^{\alpha_1 t} \nabla q(t)\|.$$

Therefore, using previous Lemmas 3.5, 3.6 we obtain from (3.56)

$$(3.58) \quad \tau^\beta(t) \|e^{\alpha_1 t} q\|_{H^1/\mathbb{R}}^2 \leq \tau^\beta(0) e^{-2\delta_0 t} (\|\mathbf{z}_0\|^2 + \kappa \|\nabla \mathbf{z}_0\|^2 + \kappa \|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M_1).$$

This concludes the proof. \square

Below, we prove one of the main theorem of this paper

Theorem 3.1. *Under the assumption (A1), let $\limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\mathbf{F}(t)\|^2 \leq M$, and $\mathbf{z}_0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that*

$$(3.59) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \left(\|\mathbf{z}(t)\|_{\mathbf{H}^2}^2 + (\|\mathbf{z}_t(t)\|^2 + \kappa \|\nabla \mathbf{z}_t(t)\|^2) + \|q(t)\|_{H^1(\Omega)/\mathbb{R}}^2 \right) \leq C(M).$$

Proof. From equation (3.55), we obtain

$$\tau^\beta(t) \|e^{\alpha_1 t} \nabla q\|^2 \leq C \tau^\beta(t) e^{2\alpha_1 t} \left(\|\mathbf{z}_t\|^2 + \|\kappa \tilde{\Delta} \mathbf{z}_t\|^2 + \|\tilde{\Delta} \mathbf{z}\|^2 + \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 + \|\nabla \mathbf{z}\|^2 + \|\mathbf{F}\|^2 \right).$$

The above inequality can be rewritten as

$$(3.60) \quad \begin{aligned} \tau^\beta(t) \|e^{\alpha_1 t} \nabla q\|^2 &\leq C \left(\tau^\beta(t) \|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \tau^\beta(t) \|\kappa e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}_t\|^2 + \tau^\beta(t) \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}\|^2 \right. \\ &\quad \left. + \tau^\beta(t) \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + \tau^\beta(t) \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{F}\|^2 \right). \end{aligned}$$

Also from (2.3), it follows that

$$c\tau^\delta(t) \|e^{\alpha_1 t} q(t)\| \leq \sup_{\mathbf{v} \in \mathbf{H}_0^1} \frac{(\mathbf{v}, \tau^\delta(t) e^{\alpha_1 t} \nabla q(t))}{\|\nabla \mathbf{v}\|} \leq \frac{1}{\sqrt{\lambda_1}} \tau^\delta(t) \|e^{\alpha_1 t} \nabla q(t)\|.$$

Hence, we arrive at

$$(3.61) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} q(t)\|_{H^1/\mathbb{R}}^2 &\leq C \left(\limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{z}_t\|^2 + \limsup_{t \rightarrow \infty} \tau^\beta(t) \|\kappa e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}_t\|^2 \right. \\ &\quad \left. + \limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}\|^2 + \limsup_{t \rightarrow \infty} \tau^\beta(t) \|\mathbf{z}\|^2 \|\nabla \mathbf{z}\|^2 \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 \right. \\ &\quad \left. + \limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 + \limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} \mathbf{F}\|^2 \right) \\ &\leq C(M) + \limsup_{t \rightarrow \infty} \tau^\beta(t) \|\kappa e^{\alpha_1 t} \tilde{\Delta} \mathbf{z}_t\|^2. \end{aligned}$$

From the equation (3.5), we obtain

$$(3.62) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\kappa \tilde{\Delta} \mathbf{z}_t\|^2 &\leq C \left(\limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\mathbf{z}_t\|^2 + \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\tilde{\Delta} \mathbf{z}\|^2 \right. \\ &\quad \left. + \limsup_{t \rightarrow \infty} \tau^\beta(t) \|\mathbf{z}\|^2 \|e^{\alpha_1 t} \nabla \mathbf{z}\|^2 \right. \\ &\quad \left. + \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\nabla \mathbf{z}\|^2 + \limsup_{t \rightarrow \infty} \tau^\beta(t) e^{2\alpha_1 t} \|\mathbf{F}\|^2 \right) \\ &\leq CM. \end{aligned}$$

Hence, a use of (3.62) in (3.61) shows

$$(3.63) \quad \limsup_{t \rightarrow \infty} \tau^\beta(t) \|e^{\alpha_1 t} q(t)\|_{H^1/\mathbb{R}}^2 \leq C(M).$$

The rest of the proof follows from Lemmas 3.1, 3.2 and 3.3. □

Remark 3.1. *If the forcing function satisfies the property*

$$(3.64) \quad \limsup_{t \rightarrow \infty} t^\beta e^{2\alpha_1 t} \|\mathbf{F}(t)\|^2 = 0,$$

then as a consequence of Theorem 3.1, we obtain for $0 < \bar{t} \leq t$

$$(3.65) \quad \limsup_{t \rightarrow \infty} t^\beta e^{2\alpha_1 t} \left(\|\mathbf{z}(t)\|_{\mathbf{H}^2}^2 + (\|\mathbf{z}_t(t)\|^2 + \kappa \|\nabla \mathbf{z}_t\|^2) + \|q(t)\|_{H^1(\Omega)/\mathbb{R}}^2 \right) = 0,$$

and as $t \rightarrow \infty$

$$(3.66) \quad \left(\|\mathbf{z}(t)\|_{\mathbf{H}^2}^2 + (\|\mathbf{z}_t(t)\|^2 + \kappa \|\nabla \mathbf{z}_t\|^2) + \|q(t)\|_{H^1(\Omega)/\mathbb{R}}^2 \right) = O(t^{-\beta} e^{-2\alpha_1 t}) \quad \text{as } t \rightarrow \infty.$$

Below we prove two Theorem which are valid for all $t > 0$.

Theorem 3.2. *Under the assumption (A1), let $\tau^\beta(t)e^{2\alpha_1 t}\|\mathbf{F}(t)\|^2 \leq M$, and $\mathbf{z}_0 \in \mathbf{H}^2 \cap \mathbf{H}_0^1$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that for all $t > 0$*

$$\begin{aligned} & \tau^\beta(t)\|e^{\alpha_1 t}\mathbf{z}(t)\|_{\mathbf{H}^1}^2 + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)(\|e^{\alpha s}\mathbf{z}_t(s)\|^2 + \kappa\|e^{\alpha s}\nabla\mathbf{z}_t(s)\|^2)ds \\ & + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|e^{\alpha s}\tilde{\Delta}\mathbf{z}(s)\|^2 ds + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|q\|_{H^1(\Omega)/\mathbb{R}}^2 ds \\ & \leq C\frac{M}{2\delta_0} + C\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2). \end{aligned}$$

Proof. From Lemmas 3.2 and 3.3, we find that

$$\begin{aligned} & e^{-2\delta_0 t} \left(\int_0^t \tau^\beta(s)(\|e^{\alpha s}\mathbf{z}_t(s)\|^2 + \kappa\|e^{\alpha s}\nabla\mathbf{z}_t(s)\|^2)ds + \int_0^t \tau^\beta(s)\|e^{\alpha s}\tilde{\Delta}\mathbf{z}(s)\|^2 ds \right) \\ & \leq Ce^{-2\delta_0 t} \left(\int_0^t \tau^\beta(s)e^{2\alpha s}\|\mathbf{F}\|^2 ds + \tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2) \right. \\ & \quad \left. + \int_0^t \tau^\beta(s)\|\mathbf{z}(s)\|^2\|\nabla\mathbf{z}(s)\|^2\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds \right) \\ & \leq C \left(M\frac{1-e^{-2\delta_0 t}}{2\delta_0} + e^{-2\delta_0 t}\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2) + \int_0^t \tau^\beta(s)\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds \right) \\ & \leq C \left(M\frac{1-e^{-2\delta_0 t}}{2\delta_0} + e^{-2\delta_0 t}\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2) \right) \\ & \leq C\frac{M}{2\delta_0} + C\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2). \end{aligned}$$

Now from Lemma 3.6, it follows that

$$\begin{aligned} & e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\kappa\tilde{\Delta}\mathbf{z}_t(s)\|^2 ds \leq C \left(e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\mathbf{z}_t(s)\|^2 ds + Ce^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\tilde{\Delta}\mathbf{z}(s)\|^2 ds \right. \\ & \quad + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|\mathbf{z}(s)\|^2\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds \\ & \quad \left. + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\nabla\mathbf{z}(s)\|^2 ds + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\mathbf{F}(s)\|^2 ds \right) \\ & \leq C\frac{M}{2\delta_0} + C\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2). \end{aligned}$$

Now from equations (3.56) and (3.57) in Lemma 3.7, we obtain

$$\begin{aligned} & e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|q\|_{H^1(\Omega)/\mathbb{R}}^2 ds \leq C \left(e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\mathbf{z}_t(s)\|^2 ds + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\kappa\tilde{\Delta}\mathbf{z}_t(s)\|^2 ds \right. \\ & \quad + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\tilde{\Delta}\mathbf{z}(s)\|^2 ds + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\nabla\mathbf{z}(s)\|^2 ds \\ & \quad + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)\|\mathbf{z}(s)\|^2\|\nabla\mathbf{z}(s)\|^2\|e^{\alpha s}\nabla\mathbf{z}(s)\|^2 ds \\ & \quad \left. + e^{-2\delta_0 t} \int_0^t \tau^\beta(s)e^{2\alpha s}\|\mathbf{F}\|^2 ds \right) \\ & \leq C\frac{M}{2\delta_0} + C\tau^\beta(0)(\|\mathbf{z}_0\|^2 + \kappa\|\nabla\mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta}\mathbf{z}_0\|^2). \end{aligned}$$

The rest of the proof follows from the Lemma 3.5. □

Theorem 3.3. *Under the assumption (A1), let $\tau^\beta(t)e^{2\alpha_1 t}(\|\mathbf{F}(t)\|^2 + \|\mathbf{F}_t\|_{-1}^2) \leq M_1 \forall t \geq 0$. Then, there exists a positive constant $C = C(N, \nu, \lambda_1, \|f^\infty\|_{-1})$ such that for all $t > 0$*

$$\tau^\beta(t)e^{2\alpha_1 t}(\|\mathbf{z}(t)\|_{\mathbf{H}^2}^2 + \|\mathbf{z}_t(t)\|^2 + \|q(t)\|_{H^1(\Omega)/\mathbb{R}}^2) \leq \tau^\beta(0)e^{-2\delta_0 t}(\|\mathbf{z}_0\|^2 + \kappa\|\nabla \mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M_1),$$

and

$$\tau^\beta(t)e^{2\alpha_1 t}(\kappa\|\nabla \mathbf{z}_t\|^2 + \|\kappa\tilde{\Delta} \mathbf{z}_t\|^2) \leq \tau^\beta(0)e^{-2\delta_0 t}(\|\mathbf{z}_0\|^2 + \kappa\|\nabla \mathbf{z}_0\|^2 + \kappa\|\tilde{\Delta} \mathbf{z}_0\|^2) + C(M_1).$$

holds.

Proof. Proof follows from Lemmas 3.5, 3.6 and 3.7 . □

Remark 3.2. *Note that all results are valid uniformly for κ as $\kappa \rightarrow 0$. Therefore, present analysis provides also convergence of unsteady Navier-Stokes equation to its steady state system.*

4 Conclusion

In this article, the convergence of unsteady solution to its steady state solution is established. Both exponential and power convergence is shown for the unsteady solution. Results are valid for large time and also for all time to come under different prescribed conditions on the forcing function. Moreover, all the results are valid uniformly as $\kappa \rightarrow 0$.

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